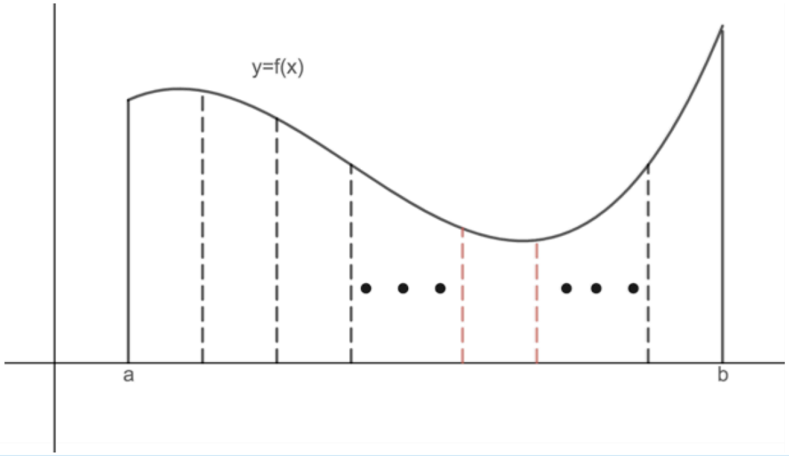
CHATPER 15 - MULTIPLE INTEGRALS

15.1 Double Integrals Over Rectangles

<u>Review 5A – The development of the definite integral</u>



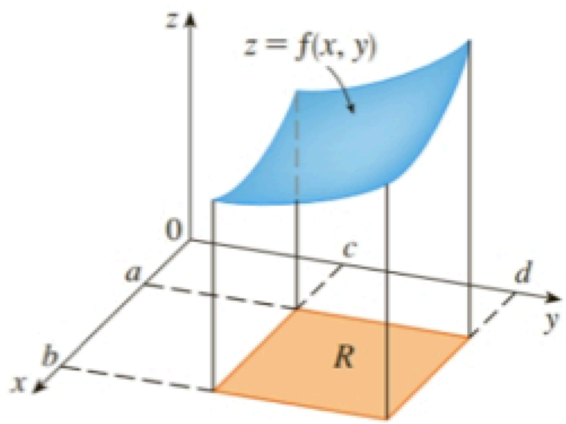
2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

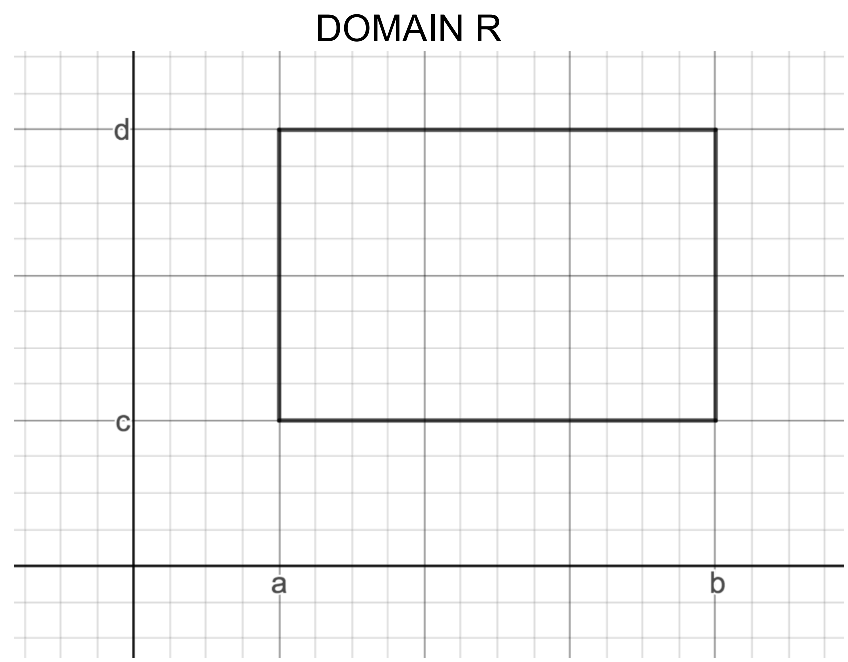
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Extend to Multivariable:

<u>Simple Case: Domain is a rectangular region. R: [a,b]X[c,d]</u>





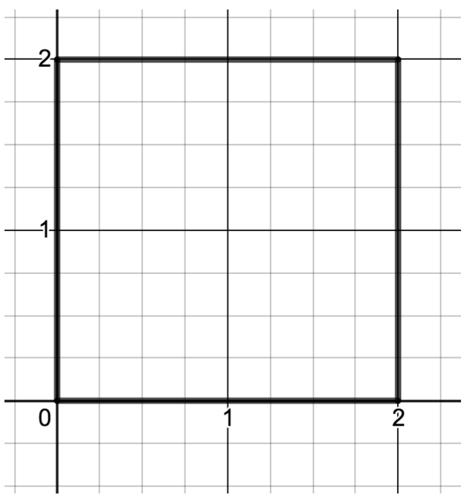
5 Definition The **double integral** of f over the rectangle R is

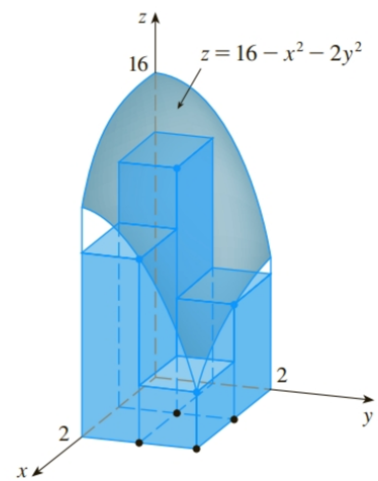
$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

How do we CALCULATE this integral? For starters, we can ESTIMATE it the way we estimated single integrals using a Riemann Sum.

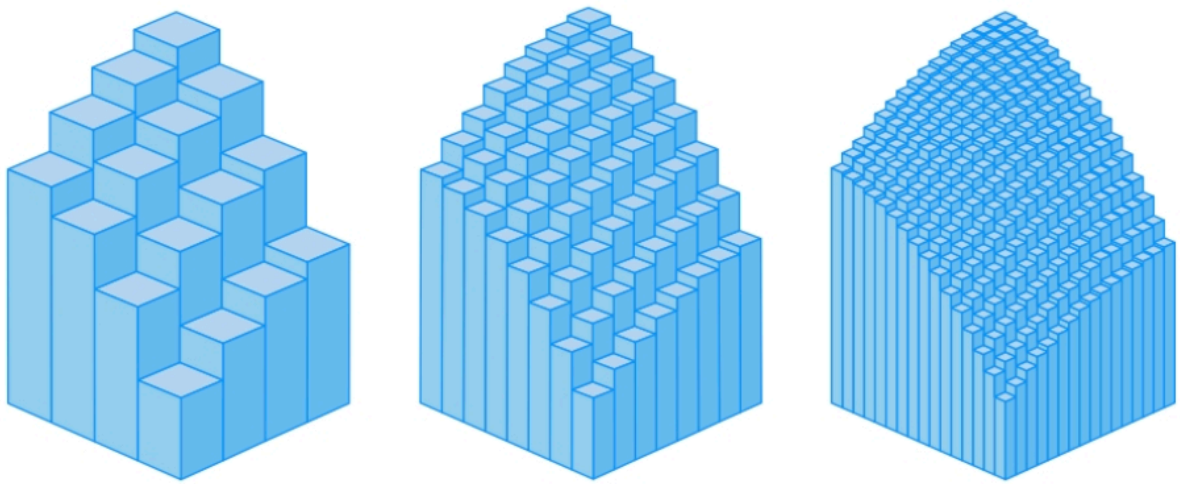
Example: Estimate $\iint_R (16 - x^2 - 2y^2)\, dA$ where R= [0,2] X [0,2] is subdivided into 4 subrectangle or equal size, and choosing the sample point to be the upper right corner of each subrectangle.





Are there applications of double integral geometrically or physically?

From the last example, if we take more and more subrectangles, this is what it would look like.



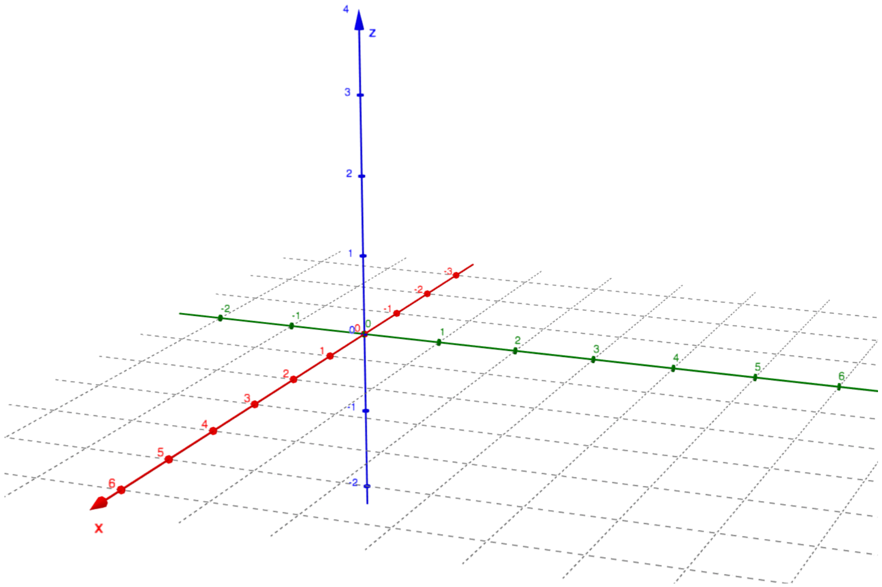
(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Using the volume interpretation to compute an integral.

Compute $\iint_R (2-2y)\,dA$ where R=[0,3]X[0,1]

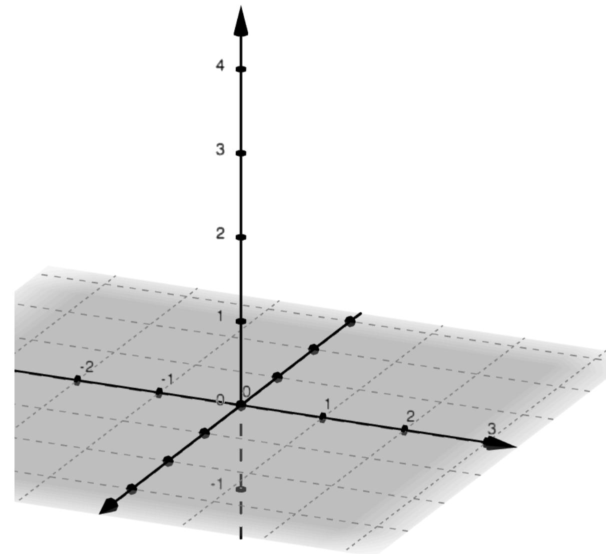


Other physical applications mass, area:

Calculating Double Integrals as an iterated integral.

Example: Calculate the volume under z=4-x-y over R: [0,2]X[0,1]

Sketch:



Use Volume by Slicing (5A: 5.2)

Case1: Take slices perpendicular to X-AXIS

$$V = \int_0^2 A(x)\,dx$$

Case 2: Take slices perpendicular to Y-AXIS

$$V = \int_0^1 A(y)\,dy$$

10 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example: Calculate $\iint_R \left(1-6x^2y\right)dA$ where R: [0,2]X[-1,1]

Example: Sometimes, choice of order matters: Calculate $\iint_R y\sin(xy)\,dA$ where R: [1,2]X[0, π]

Review of 15.1

Last time, we defined the double integral of f(x,y) over a simple, rectangular region.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

What does it mean?

If f(x,y)>0 then the double integral gives the volume under f(x,y) over R.

If f(x,y) = 1, then the double integral gives the area R.

If $f(x,y)\Delta A$ has physical meaning (like mass per unit area times area) then the double integral is the total of that physical quantity (like total mass of R)

How do we compute it?

10 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

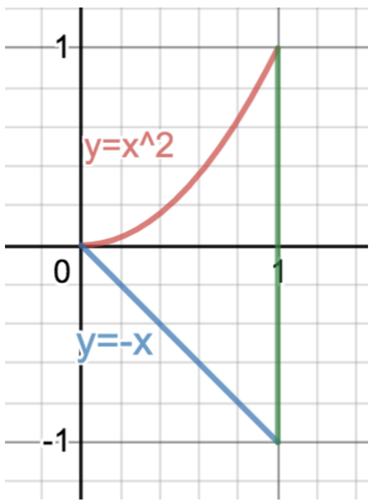
From HW 15.1

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi \, d\phi \, dt$

Tool for double integrals in special case that f(x,y) can be written as g(x)h(y)…caution

$$\boxed{11} \quad \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

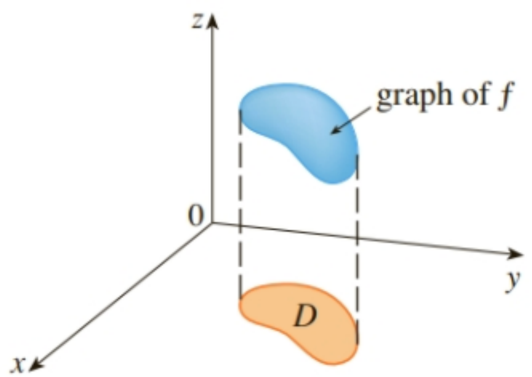
5A review problem: Find the area between y=-x and $y=x^2$ over [0,1]

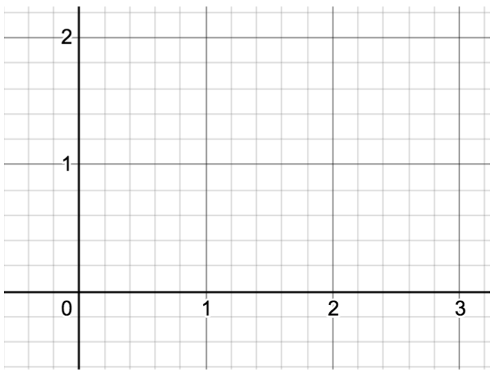


TYPE 1: dx

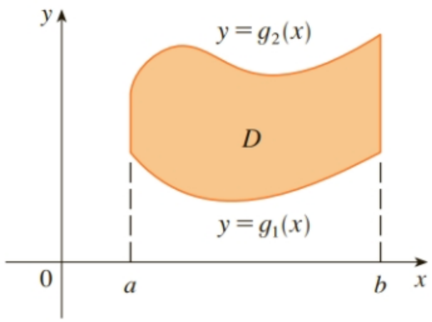
TYPE 2: dy

Given f(x,y) defined over a non-rectangular region D:





If D is TYPE 1 Region

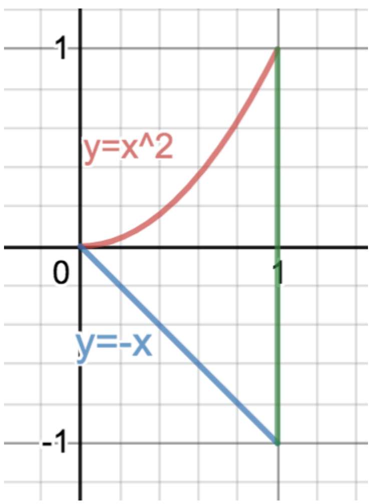


3 If f is continuous on a type I region D such that

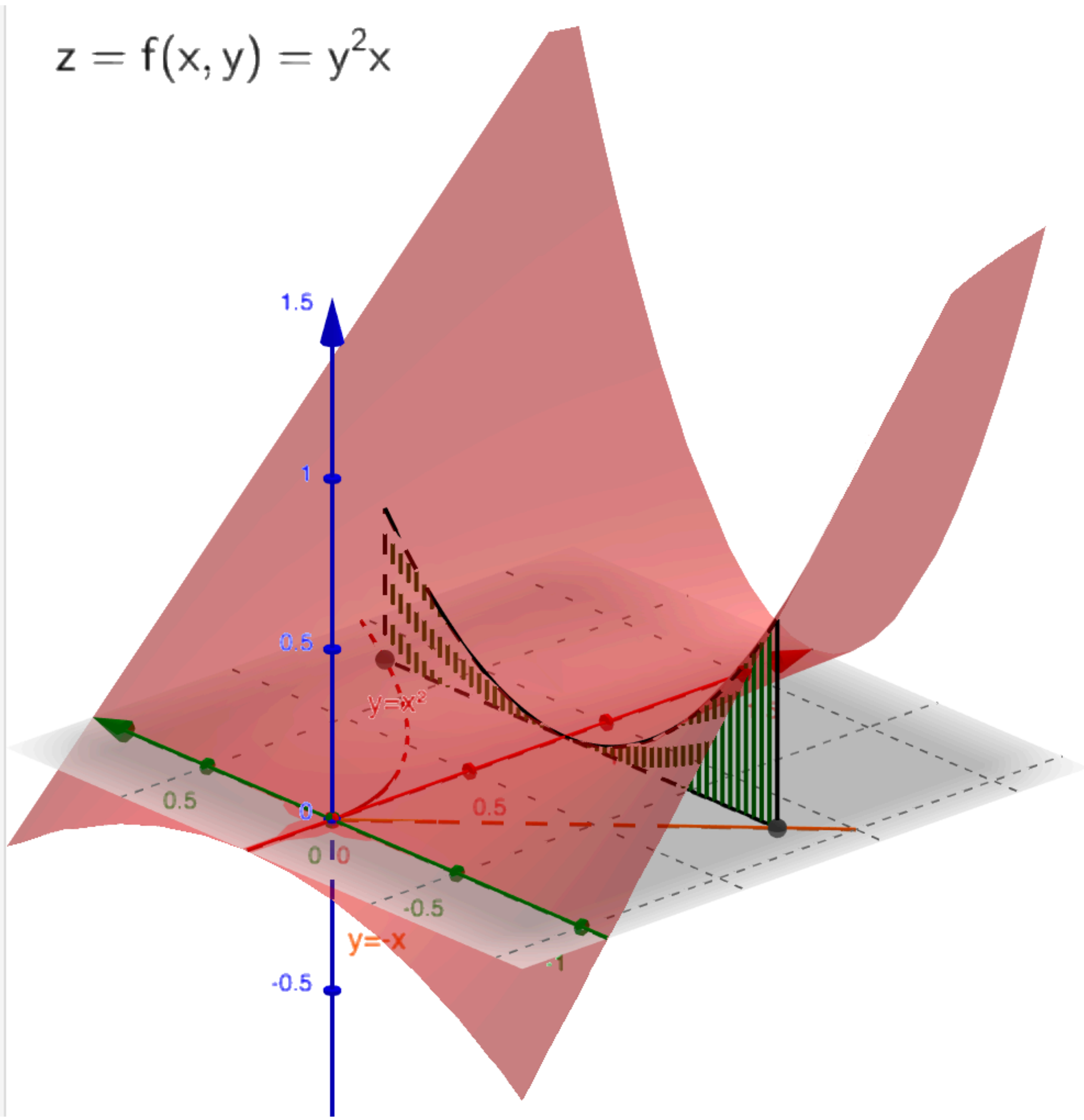
$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Example: Evaluate $\iint_D xy^2\, dA$ where D is given as shown

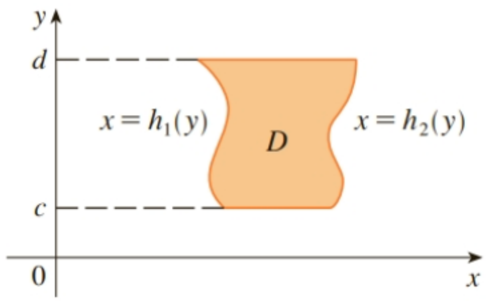


See double integral example on 5C page: https://www.geogebra.org/m/ypbjEFuv



$z = f(x, y) = y^2x$
1.5
1
0.5
0
-0.5
$y=x^2$
0.5
0.5
0
0
-0.5
y=-x

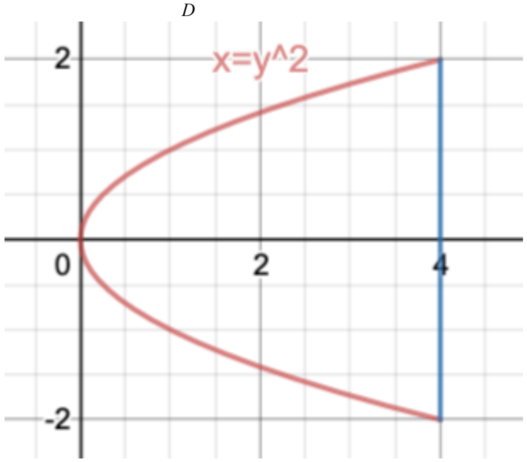
If D is TYPE 2 Region:



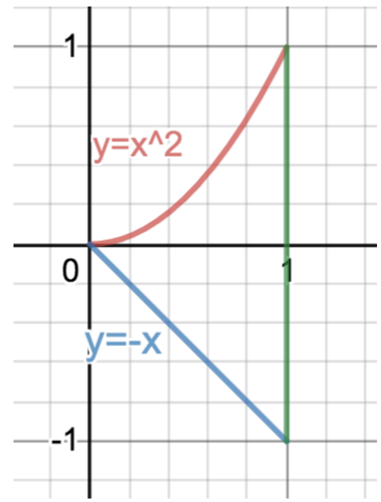
5 $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

Example: $\iint_D (x+y)dA$ where D is shown below.



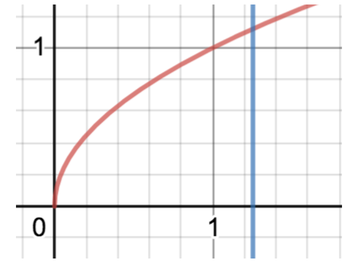
Redo previous example, with D as a TYPE 2 region: $\iint_D xy^2\, dA$ where D is given as shown



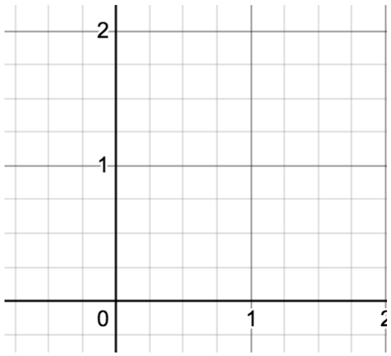
The order we choose to integrate depends on ______________________________ and ______________________.

Sometimes there is a clearly better choice.

Example: Evaluate $\iint_D y\cos(x^2)\,dA$ where D is the region enclosed by $y=\sqrt{x}$, $x=\sqrt{\frac{\pi}{2}}$, and the x axis.

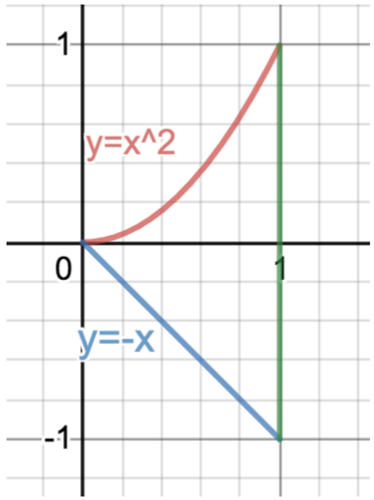


Example: Changing the order of integration – recreating the domain. Evaluate: $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$



Using double integrals as AREA:

Use double integrals to find the area between y=-x and y=x^2 over [0,1]

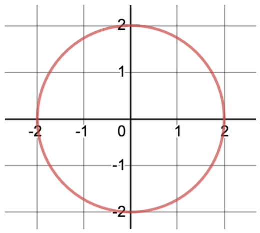


Often, this idea is used backwards:

Compute $\iint_D 4\,dA$ where D is the region contained in $x^2 + y^2 = 25$

15.3 Double Integrals In Polar Coordinates

Motivation: Evaluate: $\iint_D \sqrt{x^2+y^2}\, dA$ where D is a circle of radius 2, centered at the origin.



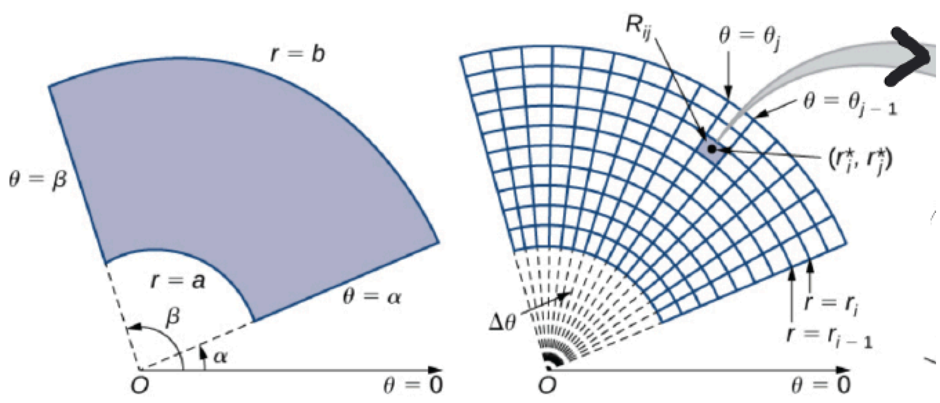
Recall Polar Coordinates: 10.3:

$x = r\cos\theta \qquad x^2+y^2=r^2$

$y = r\sin\theta \qquad \tan\theta = \dfrac{y}{x}$

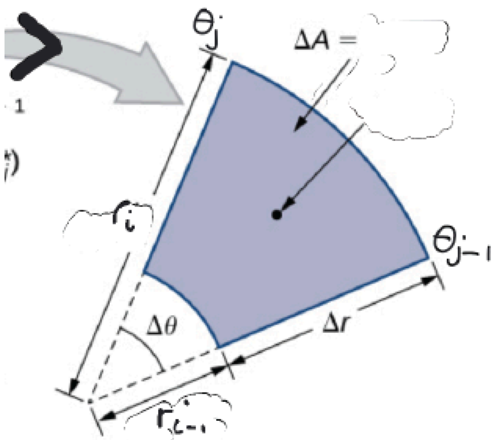
Development of Double Integral of Simple Region - "Polar Rectangle"

Given $f(x,y)$ defined over region R= $\{(r,\theta): \quad a \le r \le b,\ \alpha \le \theta \le \beta\}$



What is ΔA_{ij}?

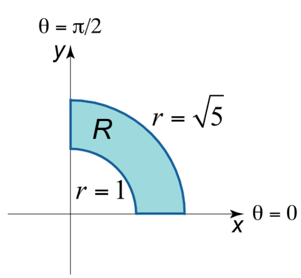
What is ΔA_{ij}?



2 Change to Polar Coodinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

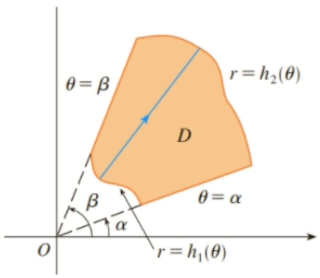
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example: where D is the region in the first quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$



Example: Evaluate: $\iint_D \sqrt{x^2+y^2}\,dA$ where D is a circle of radius 2, centered at the origin.

Extending the concept of double integral to a more complicated polar region:

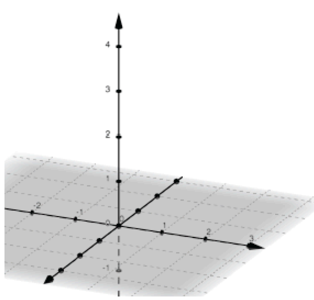


3 If f is continuous on a polar region of the form

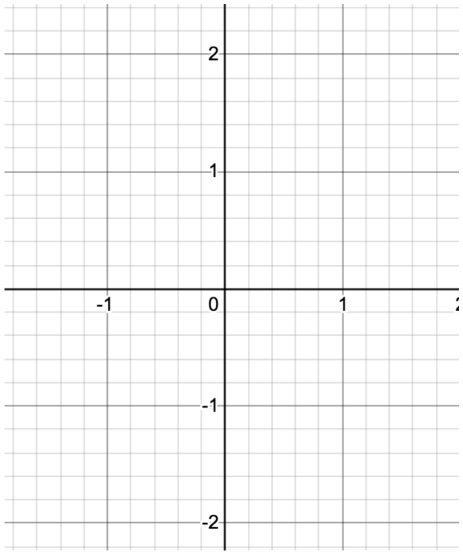
$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example: Find the volume of the solid that lies under the cone $z=\sqrt{x^2+y^2}$, above the xy plane and inside the cylinder $x^2+y^2=2y$.

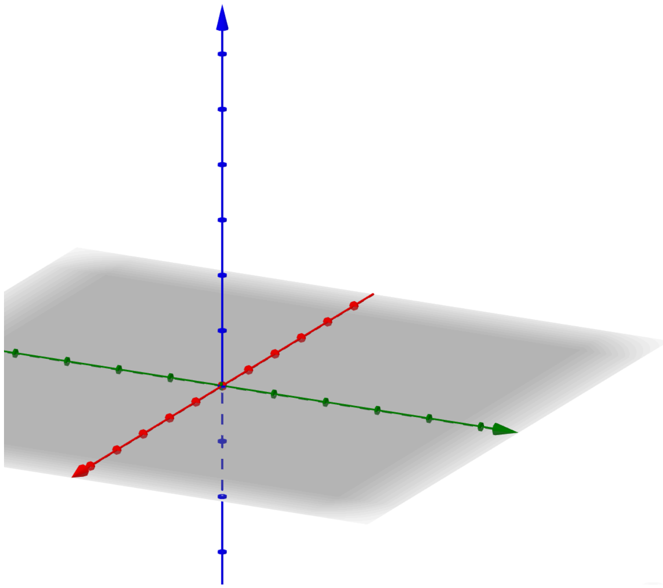


Example: Use a double integral to find the AREA of the region enclosed by $r = 1 - \sin\theta$



Extend to Multivariable: , defined over some solid E in R^3.

Simple Case: Domain is a rectangular box. B: [a,b] X [c,d] X [r,s]

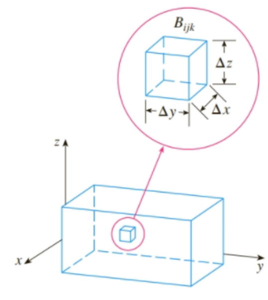


Partition [a,b] into l subintervals of equal width
Partition [c,d] into m subintervals of equal width
Partition [r,s] into n subintervals of equal width

Consider typical "sub-box"

Choose arbitrary point in sub-box:

Form product:



Sum over all sub-boxes.

Where

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Application: If f(x,y,z)=1, give the volume.

If has physical meaning, gives total…..

From 5C page under Types of Integrals:

		INTEGRALS			
			Applications		
Function	Domain	Integral	If f >0	If f=1	If f is density
f(x)	Interval [a,b] R	$\int_a^b f(x)\,dx$	Area under f	length [a,b]	mass of wire [a,b]
f(x,y)	Region D in R^2	$\iint_D f(x,y)\,dA$	Volume under f	Area of D	mass of lamina D
f(x,y,z)	Solid E in R^3	$\iiint_E f(x,y,z)\,dV$		Volume of E	mass of solid E

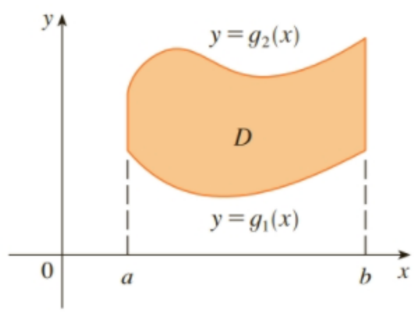
So if f > 0, Area can be computed by: ________________________ or _____________________________ and volume can be computed by ____________________________ or ______________________________________

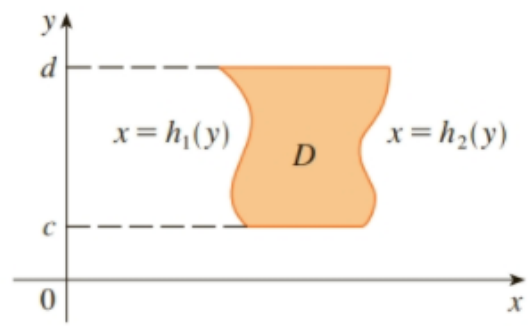
We compute triple integrals as an iterated integral. Note: There are SIX possible orders of evaluation.

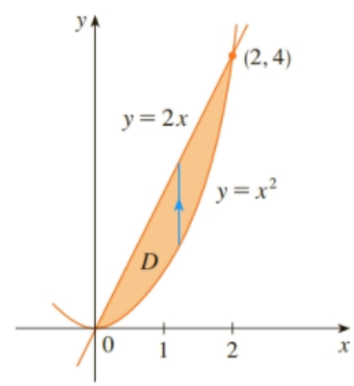
Example: for B=[0,1]X[0, π/2]X[0,3]

Triple Integrals over non-rectangular solids:

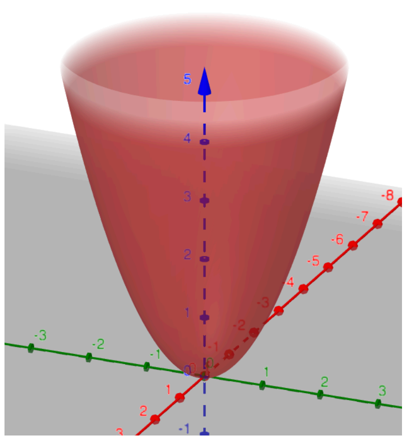
<u>Recall "type 1 region" vs type 2 region in R^2</u>

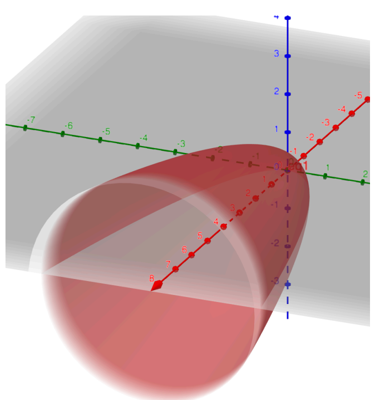


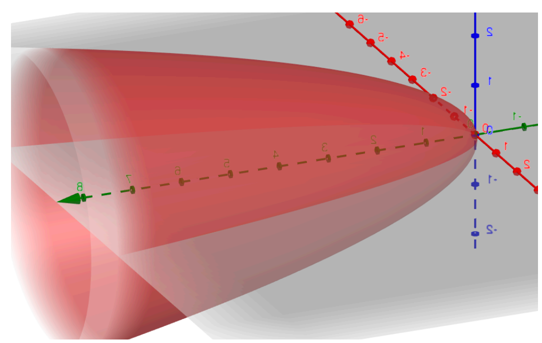




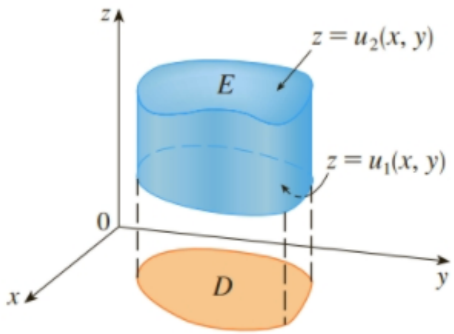
Also recall various function orientations in R^3





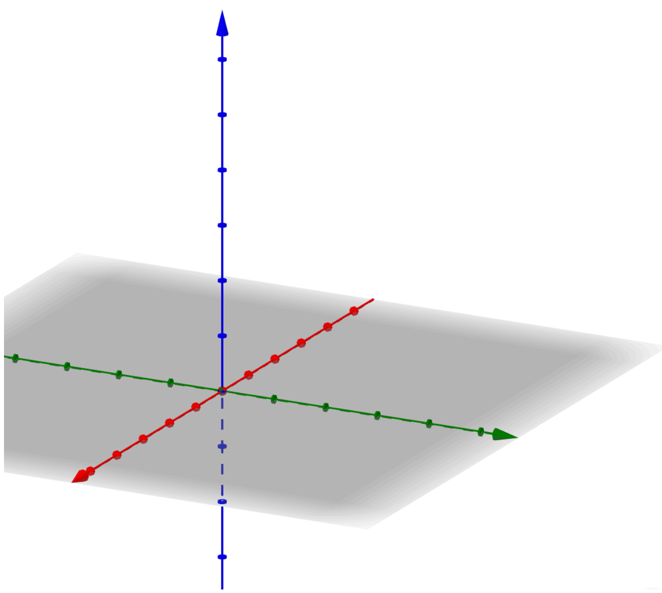


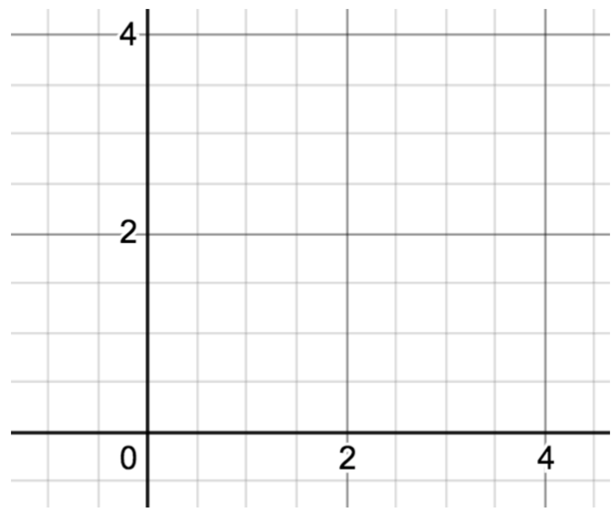
Given: $z = f(x,y,z)$, defined over some solid E in R^3.



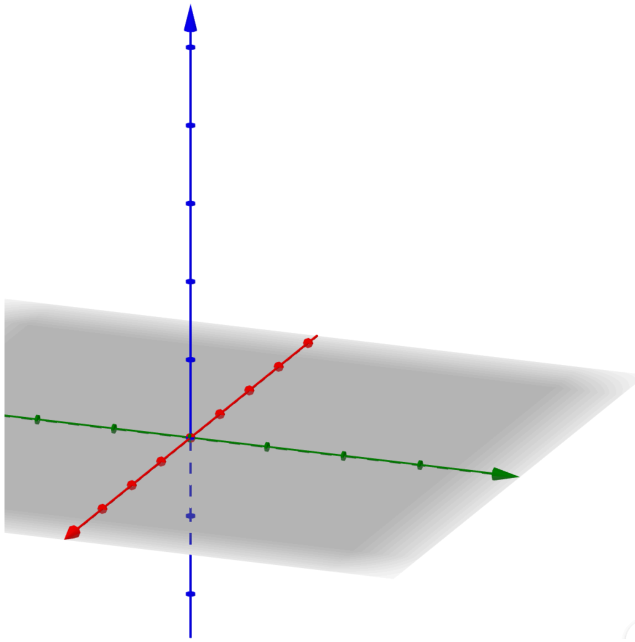
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

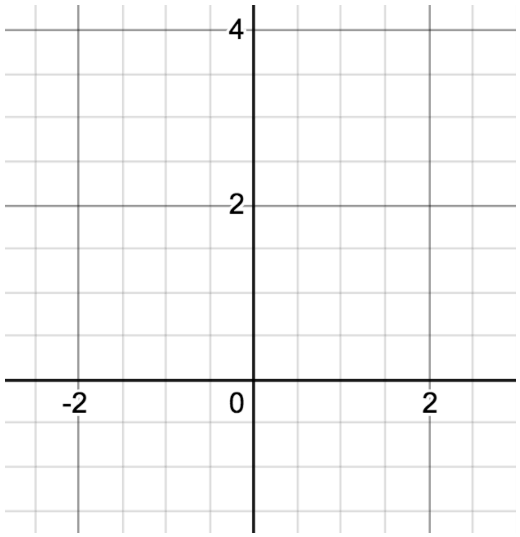
Example #1: Evaluate $\iiint_E 2x\, dV$ where E is the solid bound by 2x+3y+z=6 and the coordinate planes.





Example #2: Find the volume of the solid bound by $y = x^2$, $z = 0$, $y + z = 4$





See Geogebra Animation https://www.geogebra.org/m/akme6U7F
Recall: We can also find volume using double integrals.

Other orientations for Solid E:

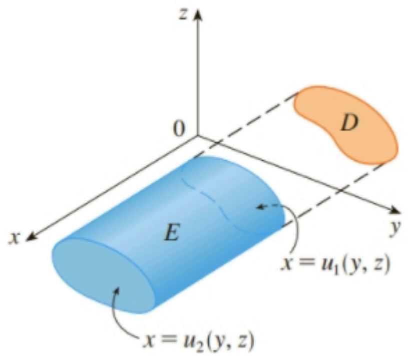


FIGURE 7
A type 2 region

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

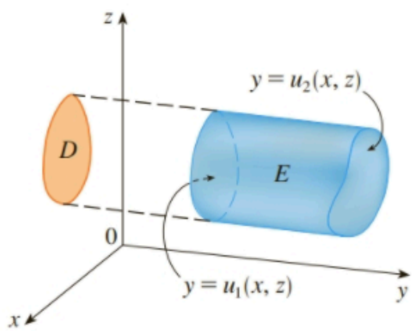
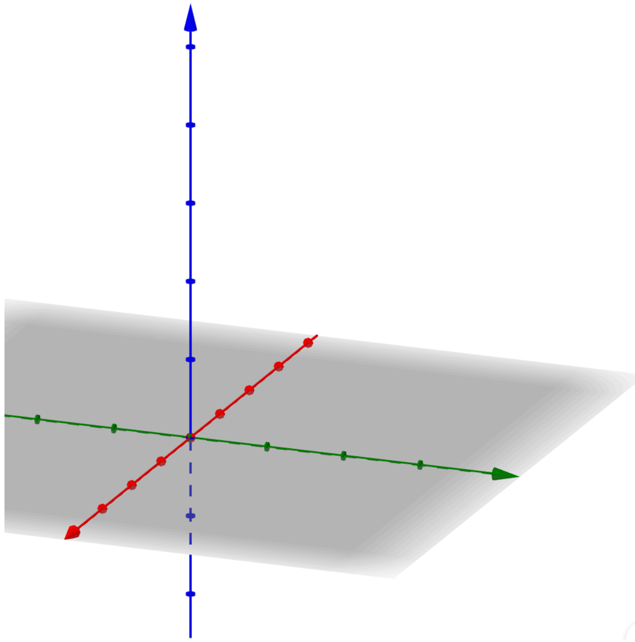


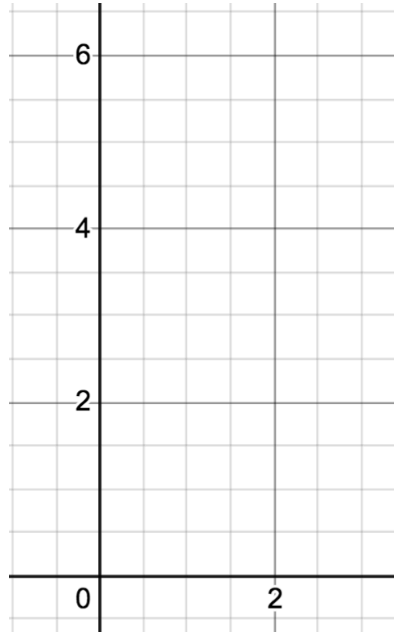
FIGURE 8
A type 3 region

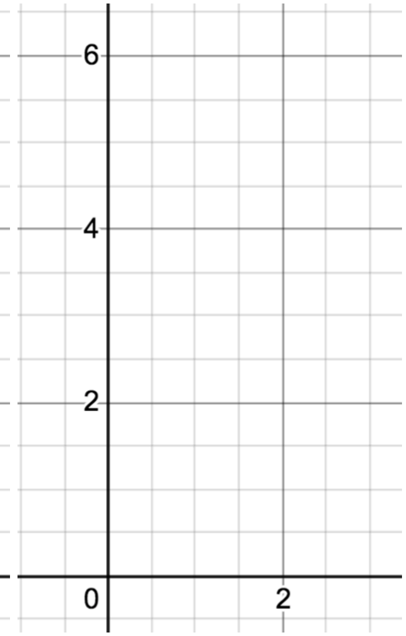
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

Redo Example #1 from different orientations

Evaluate $\iiint_E 2x\,dV$ where E is the solid bound by 2x+3y+z=6 and the coordinate planes.



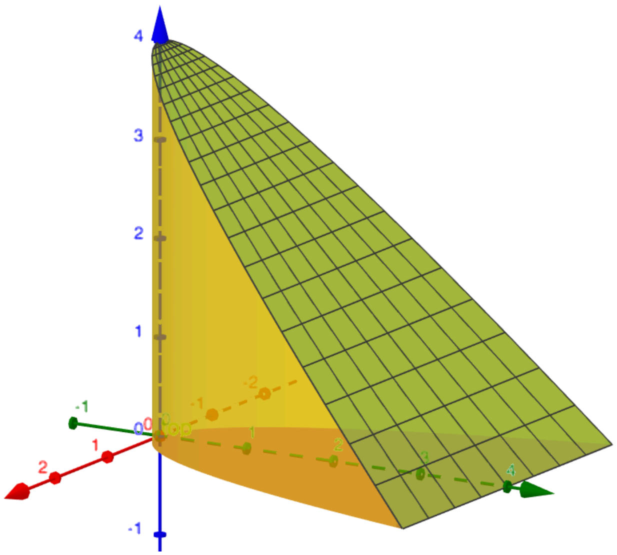


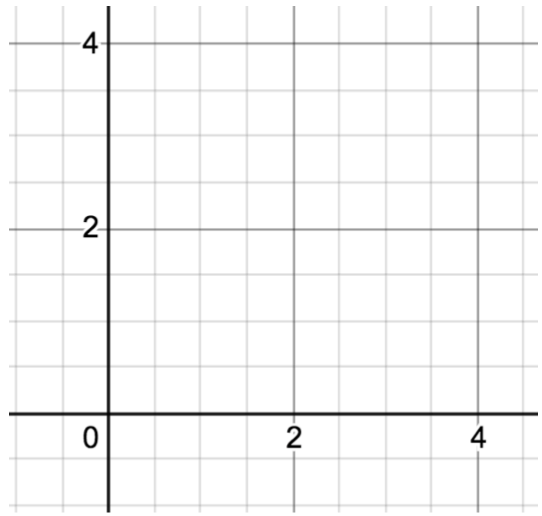


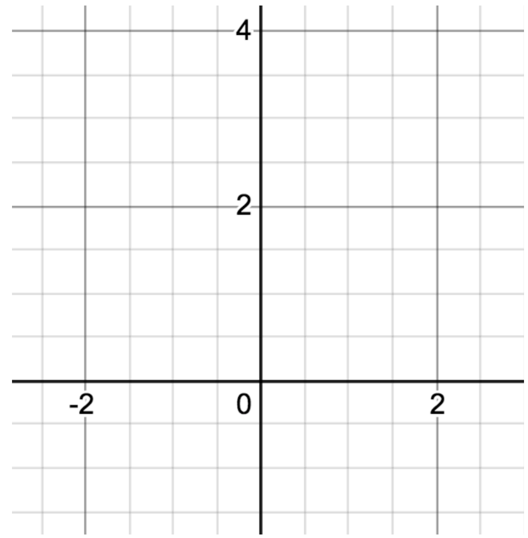
The order we choose to integrate depends on ________________________- and _________________________

Redo Example #2 from different orientations

Find the volume of the solid bound by $y = x^2, \quad z = 0, \quad y + z = 4$







Example #3:

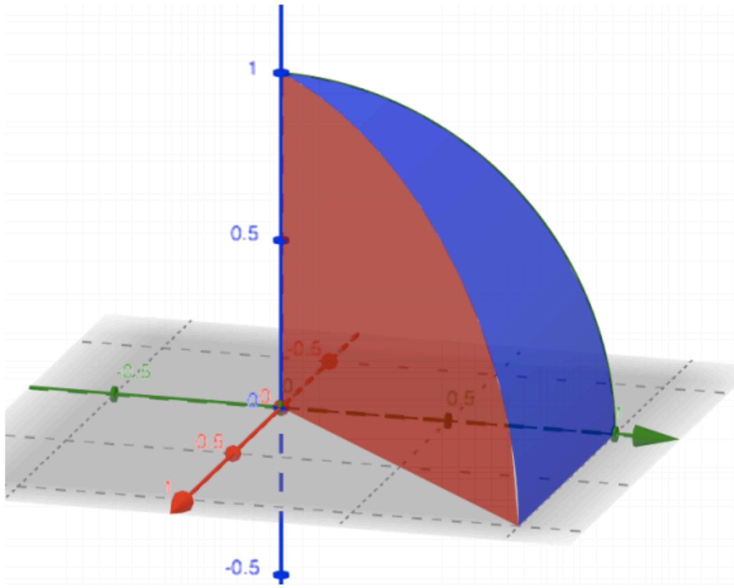
Compute $\iiint_E \left(\frac{1}{9} - z\right) dV$ where E is the solid bound by $\begin{cases} y^2 + z^2 = 9 \\ y = 3x \end{cases}$ in the first octant.

Sketch E: Link to graph on 5C page: https://www.geogebra.org/m/v8tJbE3j (Scale off here)

What do the projections look like?

THINK about it….. which order of integration might be easer? Harder?

Example #3 cont'd

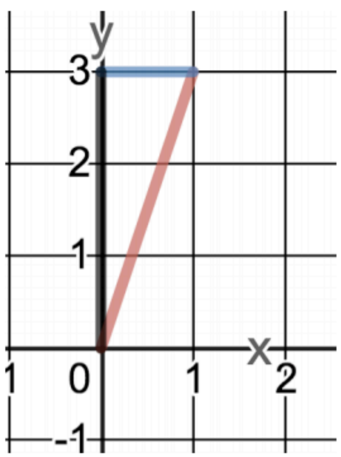


$y^2+z^2=9$

$y=3x$

<u>dz first</u>

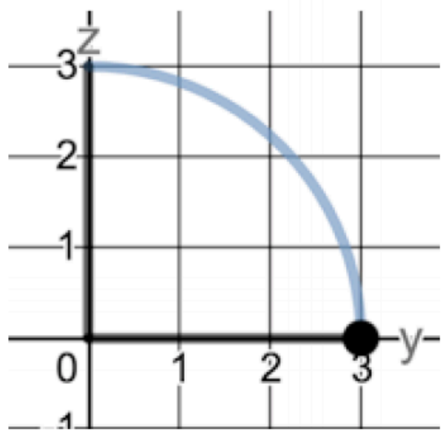
D is projection in xy plane:



$y=3x$, $y=3$

<u>dx first</u>

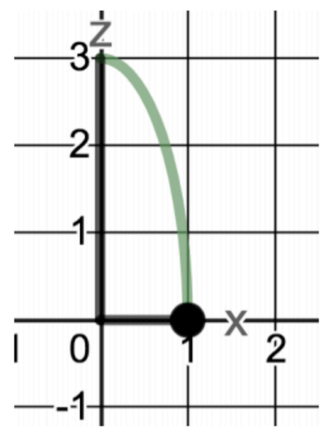
D is proj in yz plane:



$y^2+z^2=9$

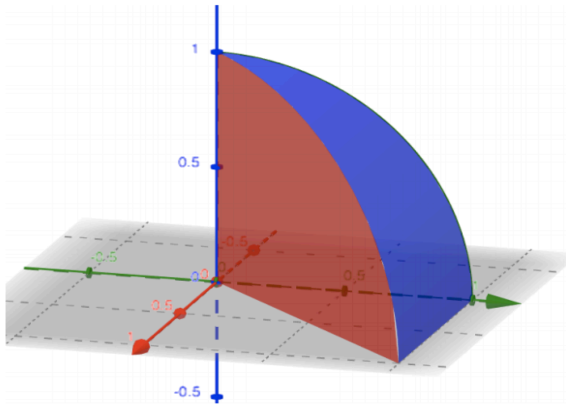
<u>dy first</u>

D is proj in xzplane:



$9x^2+z^2=9$

Example #3 cont'd

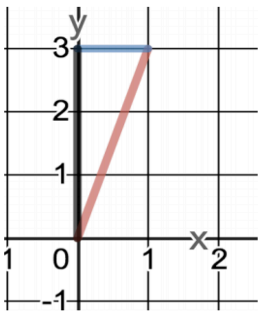


$y^2+z^2=9$

$y=3x$

dz first

D is projection in xy plane:



$y=3x$, $y=3$

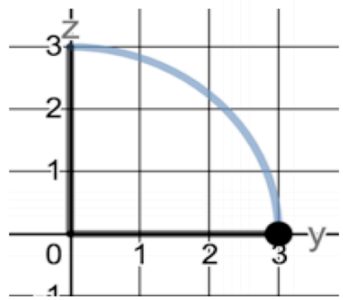
$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}\left(\frac{1}{9}-z\right)dz\,dy\,dx$$

$$\int_0^3\int_0^{\frac{y}{3}}\int_0^{\sqrt{9-y^2}}\left(\frac{1}{9}-z\right)dz\,dx\,dy$$

Answer: -73/24

dx first

D is proj in yz plane:



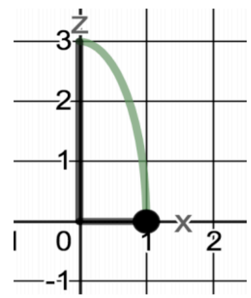
$y^2+z^2=9$

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{\frac{y}{3}}\left(\frac{1}{9}-z\right)dx\,dz\,dy$$

$$\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{\frac{y}{3}}\left(\frac{1}{9}-z\right)dx\,dy\,dz$$

dy first

D is proj in xzplane:



$9x^2+z^2=9$

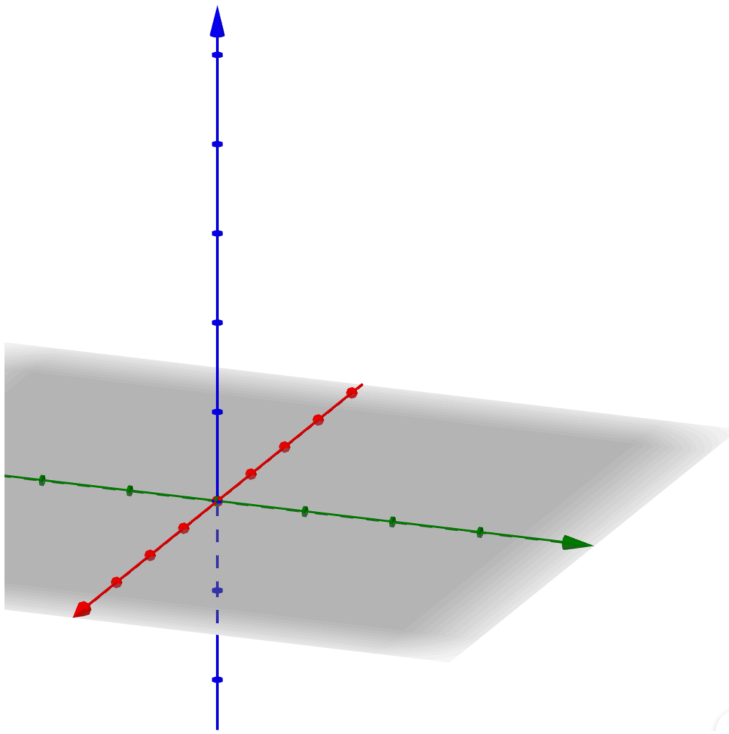
$$\int_0^1\int_0^{\sqrt{9-9x^2}}\int_{3x}^{\sqrt{9-z^2}}\left(\frac{1}{9}-z\right)dy\,dz\,dx$$

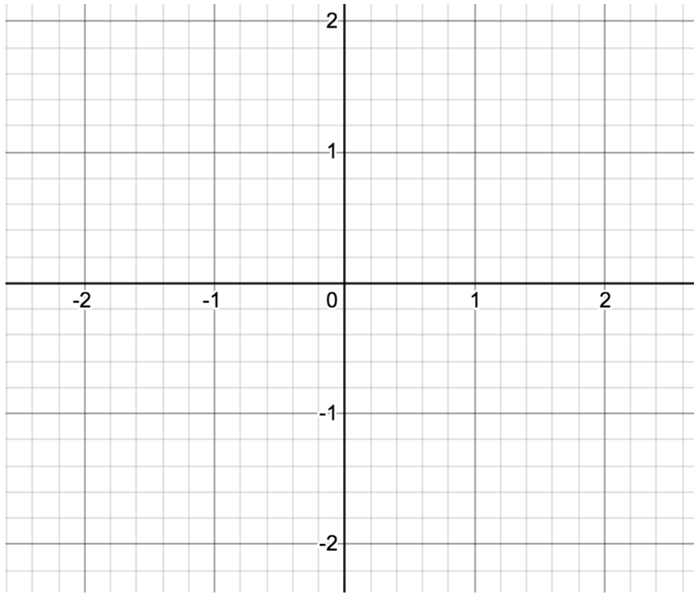
$$\int_0^3\int_0^{\sqrt{1-\frac{1}{9}z^2}}\int_{3x}^{\sqrt{9-z^2}}\left(\frac{1}{9}-z\right)dy\,dx\,dz$$

Example:

27–28 Sketch the solid whose volume is given by the iterated integral.

28. $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$

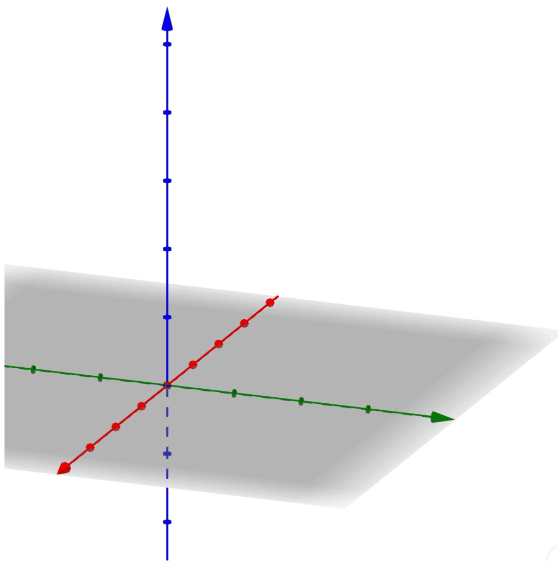




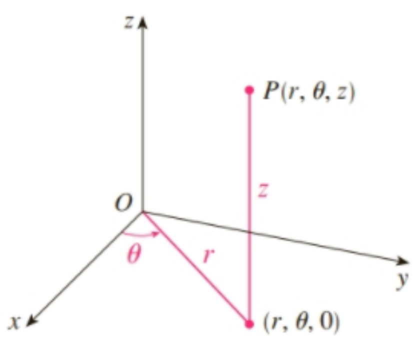
15.7 Triple Integrals in Cylindrical Coordinates

Example (Lead in to 15.7)

Find the volume inside of the cone



Cylindrical Coordinates (especially useful for circular cylinders and cones)

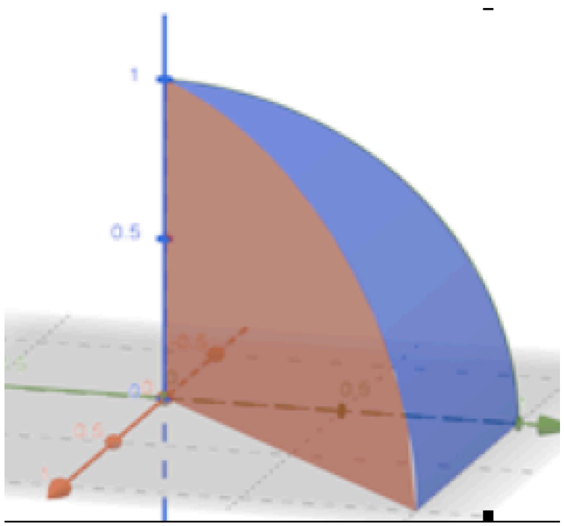


Example: Switch the following integral to Cylindrical Coordinates

Cylindrical coordinates from other orientations.

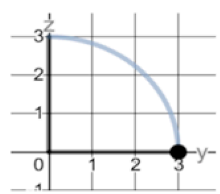
Revisit Example #3 from the previous section:

Compute $\iiint_E \left(\frac{1}{9}-z\right)dV$ where E is the solid bound by $\begin{cases} y^2+z^2=9 \\ y=3x \end{cases}$ in the first octant.



dx first

D is proj in yz plane:



$y^2+z^2=9$

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{\frac{y}{3}}\left(\frac{1}{9}-z\right)dx\,dz\,dy$$

$$\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{\frac{y}{3}}\left(\frac{1}{9}-z\right)dx\,dy\,dz$$

15.8 Triple Integrals in Spherical Coordinates

Spherical Coordinates (especially useful for spheres and cones)

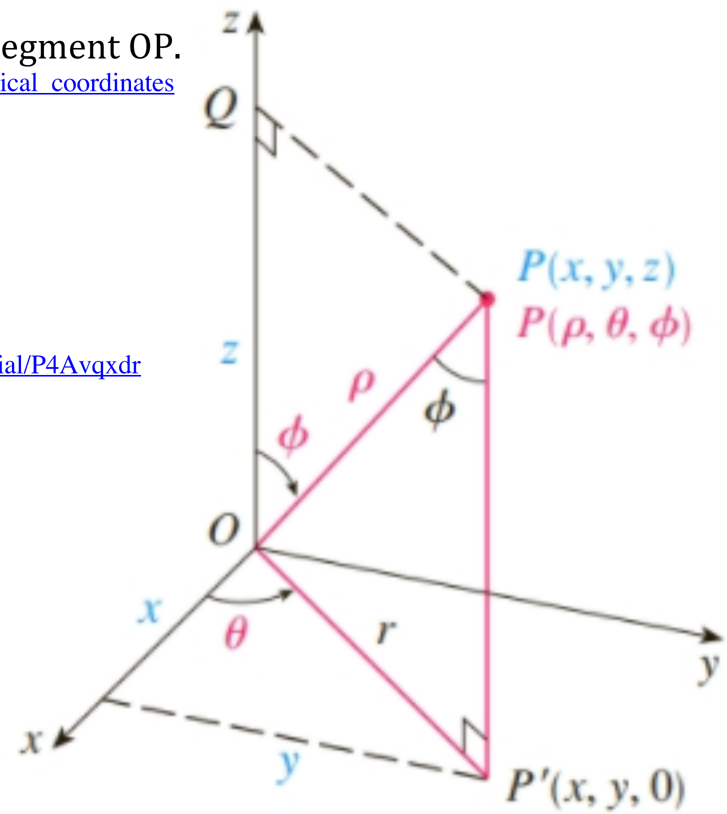
The point P can be exressed as (ρ, θ, ϕ) where:

ρ: distance from the origin to P

θ: as before

ϕ: the angle between the positive z axis and the line segment OP.

See on 5C page- Spherical Coordiate Animations https://mathinsight.org/spherical_coordinates



Basic Graphs:

$\rho=\rho_o$

$\phi=\phi_o$

See on 5C page- Simple spherical solids https://www.geogebra.org/m/RtISr7GW#material/P4Avqxdr

Derivation of Conversion Equations:

Example: Converting Coordinates of Points

Convert from Rectangular to Spherical

Convert from Spherical to Rectangular

Examples: Converting Equations

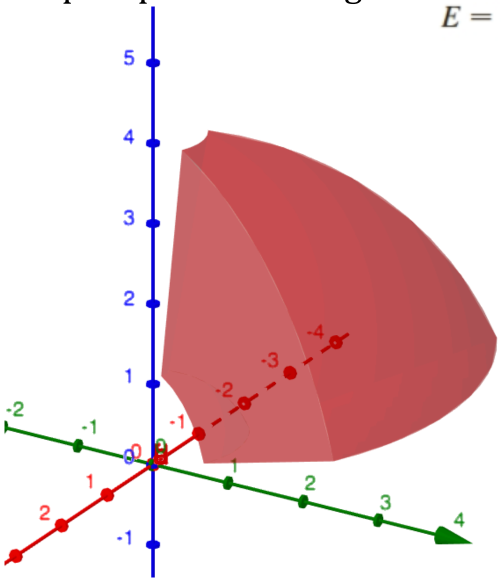
Convert the equation $z = x^2 + y^2$ to spherical coordinates.

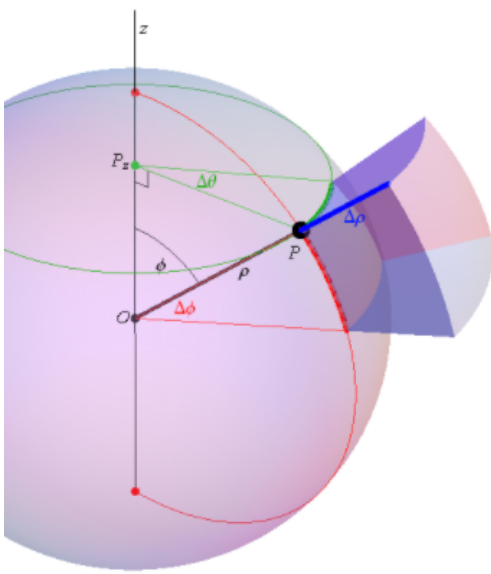
Convert the equation $\rho = 2\cos\phi$ to rectangular coordinates.

<u>Development of Triple Integral in Spherical Coordinates.</u>

Simple Spherical wedge:

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$





3

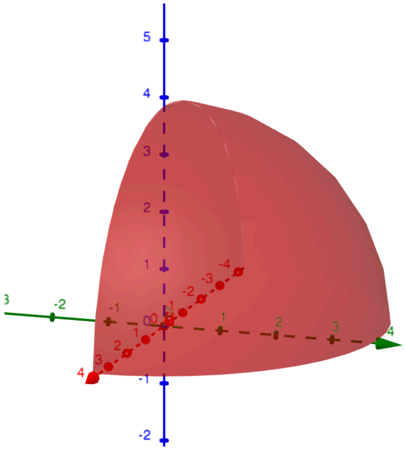
$$\iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi\, \cos\theta, \rho \sin\phi\, \sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

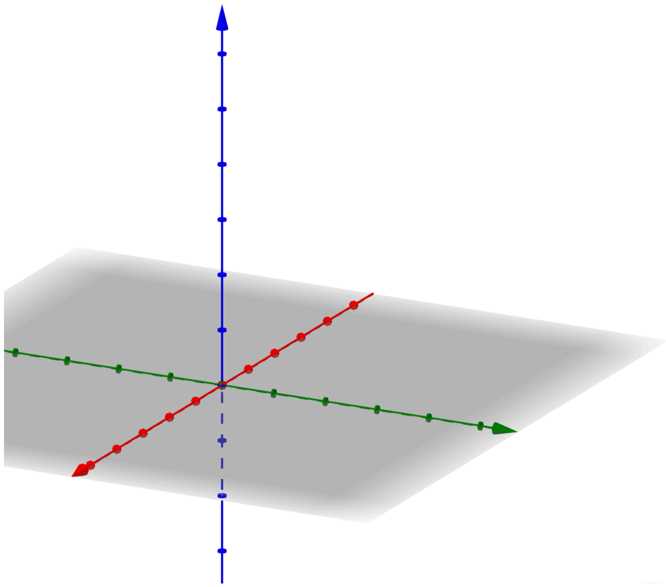
where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

Example: Evaluate $\iiint\limits_E (2-z)\,dV$ where E is the right half of the hemisphere of radius 5.

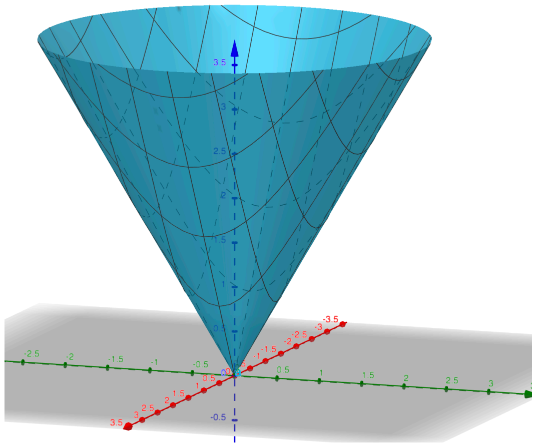


Find the volume of the solid bound by $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=z$

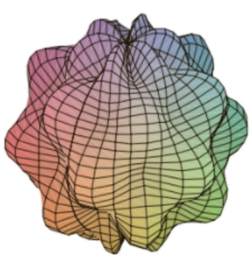


See 5C page, animation of "snow cone" https://www.geogebra.org/m/tZgrSxQ4#material/xRQ2NMMk

Revisit Previous Example: Find the volume inside of the cone $z=\sqrt{3x^2+3y^2}$; $0 \le z \le 3$



. The surfaces $\rho = 1 + \frac{1}{5}\sin m\theta \sin n\phi$ have been used as models for tumors. The "bumpy sphere" with $m = 6$ and $n = 5$ is shown. Use a computer algebra system to find the volume it encloses.



16.2i and 16.7i Two more types of integrals

So far…

New…

f(x) over interval [a,b]

f(x,y) over region D

__________f(x,y) over __________

f(x,y,z) over solid E

__________f(x,y,z) over_________

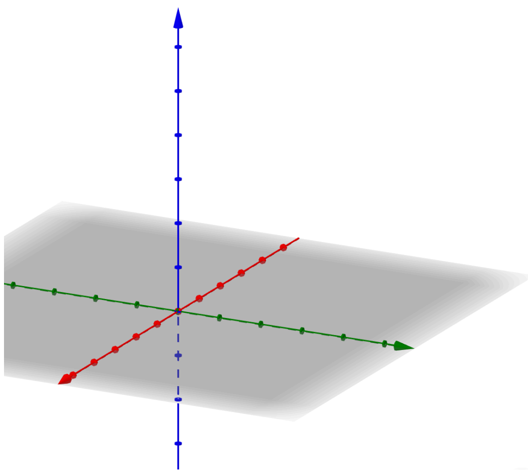
f(x,y,z) over solid E

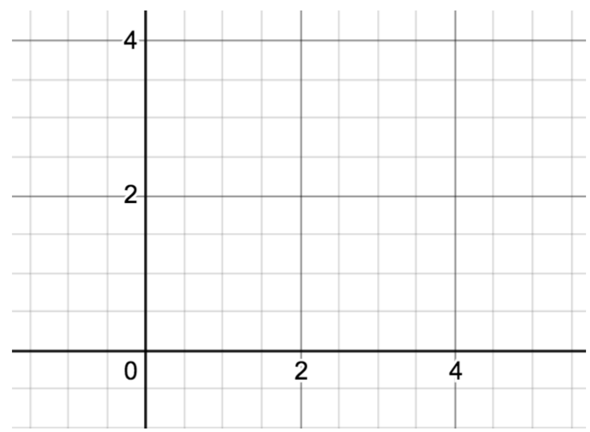
__________f(x,y,z) over_________

16.2i Line (Contour) Integrals

Development of line integral for f(x,y) in R^2 (development for f(x,y,z) in R^3 is similar).

Given f(x, y) defined over some domain D and let C, given by $\vec{r}(t) = \langle x(t), y(t) \rangle; \quad a \le t \le b$, be a smooth ($\vec{r}'(t)$ conts and $\vec{r}'(t) \ne 0$) curve in D.





Partition [a,b] into n subintervals of equal Δt

Let Pi be the point on C corresponding to $\vec{r}(t_i)$. These points break the curve into "sub-arcs".

Consider typical "sub-arc", having length ____________

Choose arbitrary point in sub-arc:

Form product:

Sum over all sub-arcs.

Where Δs_i

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

What does this mean?

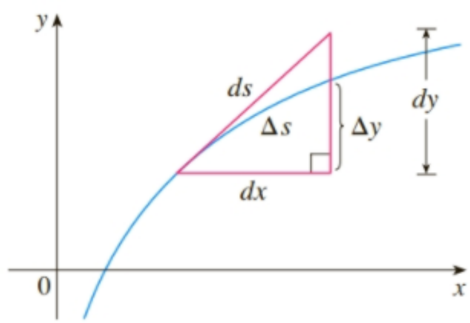
Geometric: If f(x,y)>0 (examples from 5C page)

Physical:

If f=1,

How do we compute it?

We need to get Δs_i in terms of t. In Math 5B (Section 8.1 and 10.2) we learned that



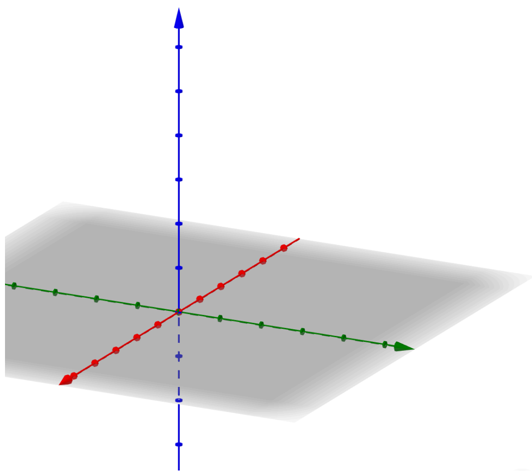
So $$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$ or $$ds = \left\|\vec{r}'(t)\right\| dt$$

And in R3 $$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

And we compute the line integral (of a scalar function with respect to arc length) by putting it all in terms of t.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example: Compute $\int_C (1-x^2)ds = \oint_C (1-x^2)ds$ where C is given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$; $0 \le t \le 2\pi$



Example: Compute $\int_C xy^2 z\, ds$ where C is the line segment from (1,0,4) to (-3,1,5).

Other line integrals: Line integrals with respect to x, y, combined

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Example: Compute $\int_C xy\,dx$ and $\int_C x^2\,dy$ where C is given by $\vec{r}(t) = \langle t, t^2 \rangle;\quad 0 \le t \le 3$

Often line integrals of this type occur together:

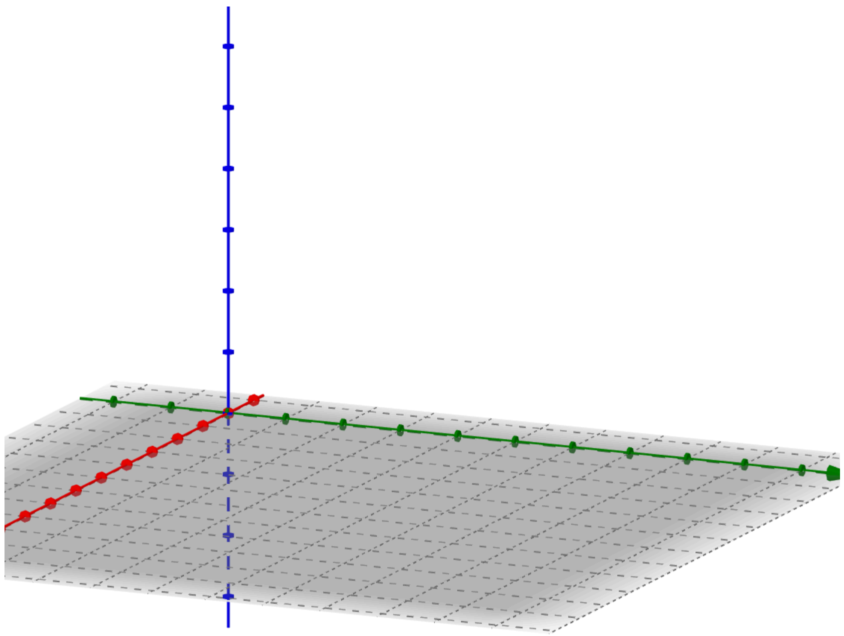
And are written in the form:

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

16.7i Surface Integrals of scalar function over surface given by a FUNCTION without parametric surfaces.

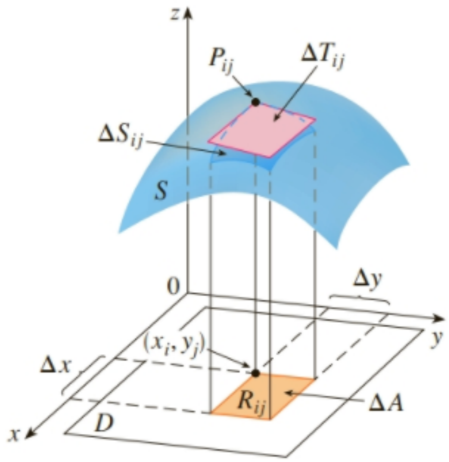
NOTE: DO NOT FOLLOW THE BOOK'S APPROACH HERE (NOR THE ONLINE SOLUTIONS). THE ONLY PART OF THE SECTION THAT WE ARE DOING AT THIS TIME IS ON PAGE 1165.

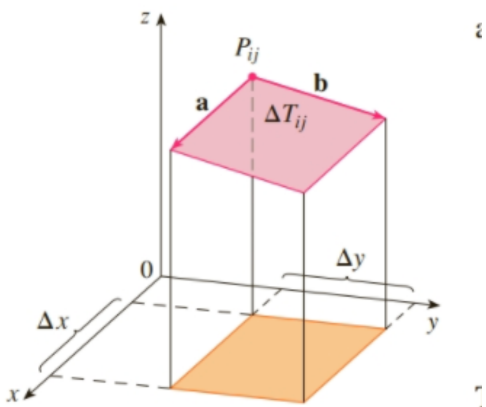
Given f(x, y, z) defined over some domain E and let S, given by z=g(x,y) over some domain D be a surface contained in E.



What is ΔS_{ij}?

Finding ΔS_{ij}, the area of the ij[th] patch. (See section 15.5)





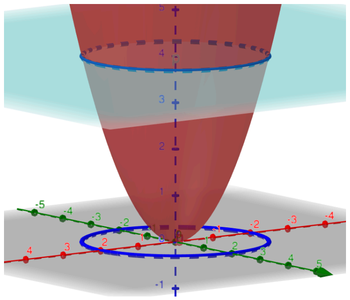
Then $\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^{\,2}+g_y^{\,2}+1}\;dA$ where dA is given by

<u>Meaning:</u>

Geometric if f(x,y,z)=1 then $\iint_S f(x,y,z)dS = \iint_S 1\,dS$

Physical:

Example: Find $\iint_S z\,dS$ where S is the part of the paraboloid $z = x^2 + y^2$ that lies under z=4.



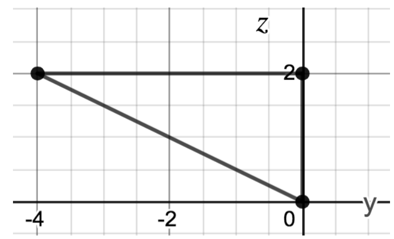
Other Orientations:

For surface S given by x=g(y,z) over a region D in the yz plane:

$$\iint_S f(x,y,z)\,dS = \iint_D f(g(y,z),y,z)\sqrt{g_y^{\,2}+g_z^{\,2}+1}\,dA$$

where dA can be viewed as dzdy, dydz or rdrdθ

<u>Example</u>: Find $\iint_S (x+3y-z^2)\,dS$ where S is the portion of $x=2-3y+z^2$ that lies over the triangle in the yz plane with vertices (0,0,0), (0,0,2) and (0,-4,2).



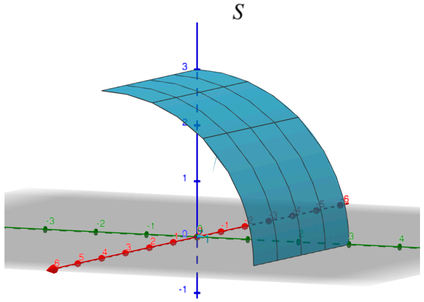
(Ans: $\dfrac{26^{3/2}-10^{3/2}}{3}$)

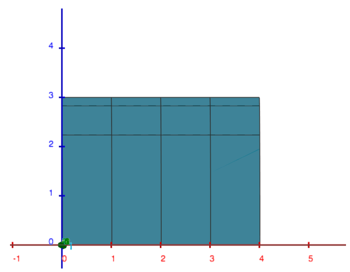
For surface S given by y=g(x,z) over a region D in the xz plane:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,g(x,z),z)\sqrt{{g_x}^2 + {g_z}^2 + 1}\,dA$$

where dA can be viewed as dzdx, dxdz or rdrdθ

<u>Example</u>: Find $\iint_S (x+z)\,dS$ where S is the part of the cylinder $y=\sqrt{9-z^2}$ that lies in the first octant between x=0 and x=4.





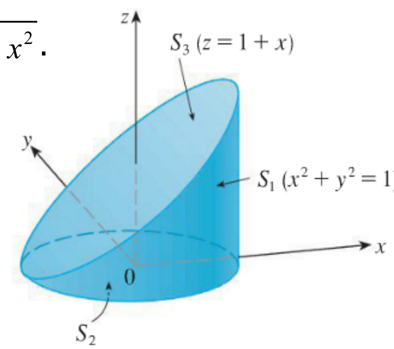
(ans: 36+12π)

Example from book without parametric surfaces: Piecewise Smooth Surface:

EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

See the book for S_2 and S_3.

For surface S_1, we would have to break it into two pieces, $y = \pm\sqrt{1-x^2}$.



S_{1a}: $y = +\sqrt{1-x^2}$

$$y = g(x,z) = \sqrt{1-x^2} \quad \Rightarrow g_x = \frac{-x}{\sqrt{1-x^2}}; \quad g_z = 0$$

$$\Rightarrow dS = \sqrt{\left(\frac{-x}{\sqrt{1-x^2}}\right)^2 + 0^2 + 1}\; dA = \cdots = \frac{1}{\sqrt{1-x^2}} dA$$

$$\iint_{S_{1a}} z\, dS = \iint_D z \frac{1}{\sqrt{1-x^2}}\, dA$$

S_{1b}: $y = -\sqrt{1-x^2}$

$$y = g(x,z) = -\sqrt{1-x^2} \quad \Rightarrow g_x = \frac{x}{\sqrt{1-x^2}}; \quad g_z = 0$$

$$\Rightarrow dS = \sqrt{\left(\frac{x}{\sqrt{1-x^2}}\right)^2 + 0^2 + 1}\; dA = \cdots = \frac{1}{\sqrt{1-x^2}} dA$$

$$\iint_{S_{1b}} z\, dS = \iint_D z \frac{1}{\sqrt{1-x^2}}\, dA$$

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi = \left(\tfrac{3}{2} + \sqrt{2}\right)\pi$$

Surface Area

If f(x,y,z)=1 then the surface integral gives us surface area. That is, the area of the surface having projection D is given by:

$$\text{Surface Area} = \iint_S 1\, dS$$

Example: Find the area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$